Geometry of cluster varieties I
David E Speyer

\[ H^0 \quad H^1 \quad H^2 \quad H^3 \quad H^4 \quad H^5 \quad H^6 \]

\[
\begin{array}{cccccc}
1 & q^2 & q^4 & q^6 & & \\
 & & & & q^3 & \\
\end{array}
\]
First talk

- Example.
- A key tool – cluster localization.
- Example.

Second talk

- A key tool – dealing with frozen variables.
- More examples.
- Mixed Hodge structure.
Initial example:

Initial quiver:

\[ x \rightarrow [y] \]
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\[ x \rightarrow \boxed{y} \]

Mutation:

\[ x' \leftarrow \boxed{y} \]
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Clusters \((x, y), (x', y)\). We have \(xx' = y + 1\).
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Cluster algebra:

\[ A = \mathbb{C}[x, x', y^\pm 1]/(xx' - y - 1) \]

Note that we invert the frozen variable \(y\).
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Cluster variety

\[ \mathcal{Y} = \text{Spec } A = \left\{ (x, x', y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* : xx' = y + 1 \right\}. \]

What does this look like?
\{(x, x', y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* : xx' = y + 1\}

One visualization: This is \{ (x, x') \in \mathbb{C}^2 : xx' - 1 \neq 0 \}:
\[ \left\{ (x, x', y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* : xx' = y + 1 \right\} \]

Another visualization: Consider the projection onto the \( y \)-coordinate. For \( y \neq -1 \), the fiber is a cylinder; for \( y = 1 \), the fiber is a pinched cylinder. Of course, there is no fiber over \( y = 0 \).

This has a deformation retract onto the subset \(|x| = |x'|, |y| = 1\).
\[ \{ (x, x', y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* : xx' = y + 1, \ |x| = |x'|, \ |y| = 1 \} \]

Cohomology: \( H^0 = \mathbb{Z}, \ H^1 = \mathbb{Z}, \ H^2 = \mathbb{Z} \).
How does the cluster structure come in?

\[ A = \mathbb{C}[x, x', y^\pm 1]/(xx' - y - 1) \]

Clusters are \((x, y)\) and \((x', y)\). Laurent phenomenon gives:

\[ A \subset \mathbb{C}[x^\pm 1, y^\pm 1] \quad A \subset \mathbb{C}[(x')^\pm 1, y^\pm 1]. \]
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Geometrically, we have two open inclusions \((\mathbb{C}^*)^2 \to \mathcal{Y}\): One by

\[(x, y) \mapsto (x, \frac{1+y}{x}, y)\]

and the other by

\[(x', y) \mapsto (\frac{1+y}{x'}, x', y).\]

Either one of these inclusions gives a decomposition \(\mathcal{Y} = (\mathbb{C}^*)^2 \sqcup \mathbb{C}\).
Note that the point $(0, 0, -1)$ is in neither torus.

The union of cluster tori is sometimes called the \textit{cluster manifold}. It is messier than the affine variety $\text{Spec } A$ because it is missing lots of low dimensional strata. We won’t discuss it here.
Other fields:

Over $\mathbb{F}_q$, we have $q^2 - (q - 1) = q^2 - q + 1$ points.

Over $\mathbb{R}$, it is natural to look at the totally positive points:
Now, onto generalities

A a cluster algebra over $\mathbb{C}$. Recall that this means that $A$ has certain elements called *cluster variables*, which are organized into sets called *clusters*.

Our cluster will have size $d = n + m$, where $m$ of the variables are *frozen* and in every cluster. We include the reciprocals of the frozen variables in $A$.

We write $\mathcal{X} = \text{Spec } A$. To be precise, we are considering the complex points of $\text{Spec } A$, topologized using the classical topology on $\mathbb{C}$. 
For each cluster \((x_1, \ldots, x_d)\), we have the Laurent phenomenon
\(A \subseteq \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]\). This gives a map from
\(\text{Spec } \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] = (\mathbb{C}^*)^d\) to \(\mathcal{X}\).

I claim that this is an open inclusion. In other words, I claim that
\(A[(x_1 x_2 \cdots x_d)^{-1}] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]\).

**Proof:** We have \(\mathbb{C}[x_1, x_2, \ldots, x_d] \subseteq A \subseteq \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]\). So
\(A[(x_1 x_2 \cdots x_d)^{-1}] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]\). □

Thus, geometrically, the open locus \(\{x_1 x_2 \cdots x_d \neq 0\}\) in \(\mathcal{X}\) is
isomorphic to \((\mathbb{C}^*)^d\). **Each cluster gives an open torus in the cluster variety.**
We would like to make larger open sets by setting some, but not all, the variables in a cluster to be nonzero. In particular, we’d like to get open covers this way.

**Reminder of basic algebraic geometry:** Let $A$ be a ring, $\mathcal{X} = \text{Spec} \ A$, and $u_1, u_2, \ldots, u_k$ functions in $A$. Then $\{u_j \neq 0\}$ is $\text{Spec} \ A[u_j^{-1}]$. The open sets $\{u_j \neq 0\}$ cover $\mathcal{X}$ if and only if $(u_1, u_2, \ldots, u_k)$ generate the unit ideal.
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Let $A$ be a cluster algebra and $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ a cluster. Let $S \subseteq \mathbf{x}$ and let $x_S = \prod_{x \in S} x$. So we’d like to understand $A[x_S^{-1}]$. Is it a cluster algebra? Can we cover $\mathcal{X}$ by simpler cluster varieties of this form?
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There is a natural candidate cluster algebra: Take the quiver for $(x_1, \ldots, x_d)$ and declare the vertices $x_i$, for $i \in S$, to be frozen. Let $A_{\mathbf{x}, S}$ be the resulting cluster algebra. We have $A_{\mathbf{x}, S} \subseteq A[x_S^{-1}]$. Very often, we have equality.
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There is a natural candidate cluster algebra: Take the quiver for $(x_1, \ldots, x_d)$ and declare the vertices $x_i$, for $i \in S$, to be frozen. Let $A_{x,S}$ be the resulting cluster algebra. We have $A_{x,S} \subseteq A[x_S^{-1}]$. Very often, we have equality.

In order to think about this, we introduce the upper cluster algebra:

$$U = \bigcap_{(x_1, \ldots, x_d) \text{ a cluster}} \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}].$$

The Laurent phenomenon says that $A \subseteq U$. Very often, $A = U$.

We can define $U_{x,S}$ similarly to $A_{x,S}$. We have

$$A_{x,S} \subseteq A[x_S^{-1}] \subseteq U[x_S^{-1}] \subseteq U_{x,S}.$$

Thus, if $A_{x,S} = U_{x,S}$, then all are equal.
\[ A_x, S \subseteq A[x_S^{-1}] \subseteq U[x_S^{-1}] \subseteq U_x, S \]

**Theorem** (Berenstein-Fomin-Zelevinsky, *Cluster Algebras III*) If there is a seed where the mutable part of the quiver is acyclic*, then \( A = U \).

**Corollary** If the restriction of \( Q \) to the vertices not in \( S \) is acyclic, then \( A[x_S^{-1}] \) is the cluster algebra \( A_x, S \).

* A quiver is called **acyclic** if it has no directed cycle.
Theorem (Muller) Suppose that we have clusters $x^1, x^2, \ldots, x^r$ and subsets $S^1, S^2, \ldots, S^r$ with $A_{x^i, S^i} = U_{x^i, S^i}$. Suppose that the open sets $\text{Spec} \ A_{x^i, S^i}$ cover $\text{Spec} \ A$. Then $A = U$.

Definition (Muller) We call $A$ locally acyclic if $\text{Spec} \ A$ is covered by finitely many $\text{Spec} \ A_{x^i, S^i}$ where the mutable part of the quiver for each $A_{x^i, S^i}$ is acyclic.

Definition (Muller) Let $Q$ be a quiver. We define an edge $i \rightarrow j$ to be a separating edge of $Q$ if there is no bi-infinite directed walk containing $i \rightarrow j$.

Theorem (Muller) If $i \rightarrow j$ is a separating edge, then $\text{Spec} \ A = \text{Spec} \ A[x_i^{-1}] \cup \text{Spec} \ A[x_j^{-1}]$.

These results are from $A = U$ for Locally Acyclic Cluster Algebras. See also Muller’s earlier paper Locally Acyclic Cluster Algebras.
The Banff algorithm

Step 0: If $Q$ has no edges, halt. In this case, $\mathcal{X}$ looks a lot like our original example $\mathcal{Y}$; see Lam-S., *Cohomology of cluster varieties. I. Locally acyclic case*, Section 7.

Step 1: If $Q$ has edges, mutate $Q$ until you find a quiver $Q'$ with a separating edge $i \to j$. In this case, you know that $\text{Spec } A = \text{Spec } A[x_i^{-1}] \cup \text{Spec } A[x_j^{-1}]$.

Step 2: Recurse on the quivers $Q' \setminus \{i\}$ and $Q' \setminus \{j\}$.

If every branch of this recursion halts, then every localization that you met along the way was a cluster algebra and you will have computed an open cover of $\mathcal{X}$ by simple spaces.
Define a cluster algebra to be **Banff** if, either $Q$ has no edges, or else $Q$ has a separating edge $x_i \to x_j$, and $A[x_i^{-1}]$ and $A[x_j^{-1}]$ are both Banff. In other words, the Banff algorithm stops.

Define a cluster algebra to be **Louise** if, either $Q$ has no edges, or else $Q$ has a separating edge $x_i \to x_j$, and $A[x_i^{-1}]$, $A[x_j^{-1}]$ and $A[(x_ix_j)^{-1}]$ are all Louise.
A final example

\[ x_1 \rightarrow x_2. \]
A final example

\[ x_1 \longrightarrow x_2. \]

This is the cluster algebra of type $A_2$ with no frozen variables.

It is \( \{ \Delta_{12} = \Delta_{23} = \Delta_{34} = \Delta_{45} = \Delta_{15} = 1 \} \) inside \( G(2, 5) \).

It is also \( \{ (\Delta_{ij}) \in G(2, 5) : \Delta_{12}\Delta_{23}\Delta_{34}\Delta_{45}\Delta_{15} \neq 0 \}/(\mathbb{C}^*)^5 \).
A final example

\[ x_1 \rightarrow x_2. \]

The edge is separating, so \( \mathcal{X} \) is covered by the localizations \( \mathcal{Y}_1 := \{ x_1 \neq 0 \} \) and \( \mathcal{Y}_2 := \{ x_2 \neq 0 \} \). These have quivers \( x_1 \rightarrow x_2 \) and \( x_1 \rightarrow x_2 \). Each localization is isomorphic to our original example, \( \mathcal{Y} \).

The intersection \( \mathcal{Y}_1 \cap \mathcal{Y}_2 \) is \( \{ x_1 x_2 \neq 0 \} \). This is the torus \( \mathbb{C}^* \times \mathbb{C}^* \).
Topologically, this is homotopic to $(S^1)^2$ with two discs glued in, bounding circles in the two directions – in other words, $S^2$. 
In terms of cohomology, we have a Meyer-Vietores sequence

\[ 0 \rightarrow H^0(\mathcal{X}) \rightarrow H^0(\mathcal{Y}_1) \oplus H^0(\mathcal{Y}_2) \rightarrow H^0(\mathcal{Y}_1 \cap \mathcal{Y}_2) \]
\[ H^1(\mathcal{X}) \leftarrow H^1(\mathcal{Y}_1) \oplus H^1(\mathcal{Y}_2) \rightarrow H^1(\mathcal{Y}_1 \cap \mathcal{Y}_2) \]
\[ H^2(\mathcal{X}) \leftarrow H^2(\mathcal{Y}_1) \oplus H^2(\mathcal{Y}_2) \rightarrow H^2(\mathcal{Y}_1 \cap \mathcal{Y}_2) \]

\[ 0 \rightarrow H^0(\mathcal{X}) \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \]
\[ H^1(\mathcal{X}) \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}^2 \]
\[ H^2(\mathcal{X}) \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \]

So \( H^0(\mathcal{X}) = \mathbb{Z}, \ H^1(\mathcal{X}) = 0, \ H^2(\mathcal{X}) = \mathbb{Z}. \)
In terms of counting points over $\mathbb{F}_q$, we have

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A teaser for a recent paper: This is a $q$-Catalan number! See Galashin and Lam *Positroids, knots, and $(q,t)$-Catalan numbers* for much more.